

ON INTEGER SOLUTIONS TO SYSTEMS OF LINEAR EQUATIONS

By

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William George Griffiths IV

I dedicate this work to my brother Sean, my parents, my sister Kathleen, and my wonderful companion, Deidre.

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A magic square is an $n \times n$ matrix with non-negative integer entries, such that the sum of the entries in each row and column is the same. We study the enumeration and P -recursivity of these in the case in which the sum along each row and column is fixed, with the size n of the matrix as the variable. A method is developed that nicely proves some known results about the case when the row and column sum is 2, and we prove new results for the case when the sum is 3.

The second part of the dissertation deals with magic cubes, about which almost nothing is known. We apply a theorem from graph theory to get an initial result in a new direction on magic cubes, that of completion. That is, after realizing the inherent connection between magic cubes and Latin Squares, we prove a condition which guarantees that a partially complete magic cube can be completed.

CHAPTER 1 INTRODUCTION

Suppose we wish to distribute fertilizer over a square plot of land. We divide our plot into n^2 subsquares by dividing it lengthwise and widthwise into n equal pieces. Then we wish to distribute our fertilizer so that each "strip" of land gets an equal amount of fertilizer, where a "strip" of land is one piece of the length by one piece of the width of the plot of land. If we begin with m units of fertilizer, then to distribute it evenly in the prescribed way, we will need to place the same amount of fertilizer in each strip. Hence, it must be that $n|m$.

If we divide our strip of land lengthwise and widthwise by 4, and we have 16 units of fertilizer, one solution to the proposed problem is seen in Figure 1-1. This is one of many possibilities for the distribution. In general, we would like to know what we can say about the number of solutions to this problem. Trivially, if each strip gets the same amount of fertilizer, we will require that, with our strip of land written as a matrix, each row and column must have $\frac{m}{n}$ units of fertilizer placed inside. If we further divide the square plot of land, there will be even more ways to distribute the fertilizer. If we increase the amount of fertilizer, how does this affect the number of solutions?

To further discuss this concept, and motivated by our pictorial example above, we make the following definition.

$$\begin{pmatrix} 0 & 2 & 0 & 2 \\ 0 & 0 & 3 & 1 \\ 1 & 2 & 0 & 1 \\ 3 & 0 & 1 & 0 \end{pmatrix}$$

Figure 1-1: One Possible Solution

$$\begin{pmatrix} 2 & 0 & 2 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

Figure 1-2: A Magic Square of Order 3 and Line Sum 4

$$\begin{pmatrix} 17 & 24 & 1 & 8 & 15 \\ 23 & 5 & 7 & 14 & 16 \\ 4 & 6 & 13 & 20 & 22 \\ 10 & 12 & 19 & 21 & 3 \\ 11 & 18 & 25 & 2 & 9 \end{pmatrix}.$$

Figure 1-3: A Magic Square with Distinct Entries

Definition 1. *A Magic Square of order n and line sum r is an $n \times n$ matrix with non-negative integer entries such that the sum of the entries in each row and column is r .*

This is our plot of land divided into n strips lengthwise and columnwise, and with $r = \frac{m}{n}$, or, in other words, n times r is the amount of fertilizer we have to distribute.

We do mention that this is not the only definition of a magic square. The oldest definition of which we are aware is that which has the additional condition of the entries of the matrix being elements of $[n^2]$, with each element of that set used only once. Also, we could require the sum of each diagonal of the square to be the same as the line sum of the magic square. For an example, see Figure 1-3. This is a special case of the magic squares that we have defined, which also satisfies the more restrictive conditions placed by the aforementioned different definition of magic squares. However, as far as this dissertation is concerned, there is nothing special about this magic square of order 5 and line sum 64.

Let $H_n(r)$ be the number of magic squares of order n with line sum r .

Magic Squares have seen some measure of attention in the last forty years. However, the subject is considerably older than that. The first nontrivial problem was solved in 1916, by P. MacMahon, who proved the following:

$$\begin{pmatrix} & a & d & r-a-d \\ r+c-(a+b+d) & b & a+d-c & \\ b+d-c & r-b-d & c & \end{pmatrix}$$

Figure 1–4: The Four Determining Elements of a Magic Square of Order 3

Theorem 1. *For any non-negative integer r , we have $H_3(r) = \binom{r+4}{4} + \binom{r+3}{4} + \binom{r+2}{4}$.*

For the original proof of the theorem, see MacMahon [9]. We remark that the proof of this theorem has been improved over the years. For two more proofs, see Bóna [4] and Stanley [14]. We include the proof of Bóna here, as background for the subject of magic squares.

Proof. A magic square of order 3 is completely determined by four of its entries: The three entries of the main diagonal and the entry located in the second column and the first row of the magic square. The remaining elements of the square will be determined as in the figure.

What remains is to determine the number of ways one can choose the a , b , c , and d so that we have a magic square. That is, in how many ways can we choose these four entries so that all entries of the magic square are non-negative? The following are necessary and sufficient conditions on a , b , c , and d which guarantee the matrix is a magic square.

$$a + d \leq r \tag{1.1}$$

$$b + d \leq r \tag{1.2}$$

$$c \leq a + d \tag{1.3}$$

$$c \leq b + d \tag{1.4}$$

$$a + b + d - c \leq r \tag{1.5}$$

Next, we partition all of the magic squares of order 3 into three disjoint subsets:

i) Those magic squares for which $a \leq b$ and $a \leq c$.

ii) Those for which $b < a$ and $b \leq c$.

iii) Those for which $c < a$ and $c < b$.

In the first case, natural numbers a , b , c , and d produce a magic square if and only if

$$a \leq 2a + d - c \leq a + b + d - c \leq b + d \leq r. \quad (1.6)$$

Thus, the choices for these four numbers are in bijection with the 4-tuples $(a, 2a + d - c, a + b + d - c, b + d)$ which satisfy (1.6). The number of these 4-tuples is $\binom{r+4}{4}$, as we can choose four elements from the set $\{0, 1, 2, \dots, r\}$ with repetition allowed.

Now we enumerate the second case, where $b < a$ and $b \leq c$. In a method similar to the first case, the inequalities, when combined, imply

$$b \leq 2b + d - c \leq a + b + d - c - 1 \leq a + d - 1 \leq r - 1. \quad (1.7)$$

This can be proved directly from the inequalities themselves together with the fact that for all integers x and y , $x < y$ if and only if $x \leq y - 1$. As above, the possible choices for the determining entries of the magic square are in bijection with the 4-tuples $(b, 2b + d - c, a + b + d - c - 1, a + d - 1)$ satisfying (1.7). The number of these is $\binom{r+3}{4}$, as we now choose four elements from the set $\{0, 1, 2, \dots, r - 1\}$, with repetition allowed.

Now, in the final case, we have $a > c$ and $b > c$. As before, we combine the useful inequalities into a chain:

$$c \leq b - 1 \leq b + d - 1 \leq a + b + d - c - 2 \leq r - 2. \quad (1.8)$$

We choose our 4-tuple from the set $\{0, 1, 2, \dots, r - 2\}$ with repetition allowed in $\binom{r+2}{4}$ ways.

Now, we have counted each magic square of order 3 and line sum r at most once, and we are done. □

Note that this theorem concerns itself with the number of magic squares for a fixed value of n , with r variable. We investigate the behavior of $H_n(r)$ with r fixed and treat $H_n(r)$ as a function of n . For now, we will see what has come before for the case of fixed n .

Anand, Dumir, and Gupta proved the following theorem about $H_n(r)$ for fixed n . For their proof, see [1]. For a far-reaching generalization of this theorem with commutative algebra, see Stanley [13].

Theorem 2. *For fixed n , $H_n(r)$ is a polynomial in r of degree $(n-1)^2$.*

With this theorem, the question of asymptotic enumeration of magic squares becomes that of the size of the leading coefficient of the aforementioned polynomial. That is, the growth of $H_n(r)$ for fixed n is dominated by this leading coefficient. Some attention could be paid here, then, to the question: How large is this coefficient? For some results in this direction, see Bóna [3].

Now, we turn our attention to the behavior of $H_n(r)$ with fixed r . What can be said about the cases of very low n ? Trivially, there exists a bijection between magic squares of order n with line sum 1 and permutations of length n . One simply numbers the rows and columns of the magic square from 1 to n , and writes, for each row, the column number in which the one appears. This gives a permutation in one line notation.

This rather simple result leads to an important property of all magic squares. We will need Philip Hall's Marriage Theorem to prove the next result, so we state it here for the sake of completeness. To state the theorem of Hall, we will first require a definition.

Definition 2. *A choice function f is a function from $\{A_1, \dots, A_n\}$ to $A_1 \cup A_2 \cup \dots \cup A_n$ such that $f(A_i) \in A_i$.*

Definition 3. *A System of Distinct Representatives (SDR) from n sets, A_1, \dots, A_n , is a choice function f such that for all $i, j \in [n]$ $f(A_i) \neq f(A_j)$.*

In the following theorem, by $|A(I)|$ we mean the cardinality of $\bigcup\{A_i : i \in I\}$. This theorem is an adaptation of Philip Hall's Marriage Theorem. For the original, see Hall [8].

Theorem 3. *Let A_1, \dots, A_n be a family of sets. Then a SDR exists for the family if and only if $|A(I)| \geq |I|$ for all $I \subset [n]$.*

Through the use of this theorem, one can establish the following lemma. This lemma is also known as the Birkhoff-von Neumann Theorem.

Lemma 1. *Any magic square of line sum r can be expressed as a sum of r permutation matrices.*

Proof. We prove this theorem by induction on r , with the case $r = 1$ being trivial. Suppose we know that any magic square of line sum $r - 1$ can be expressed as a sum of $r - 1$ permutation matrices. Then take any magic square of line sum r . If we can find a way to subtract a permutation matrix from this and have the difference be a magic square of line sum $r - 1$, we are done. Let A_i be a family of sets with

$$A_i = \{j : (i, j) \neq 0\}. \quad (1.9)$$

In other words, A_i is the set of columns whose intersection with row i is not a 0 entry of the matrix. If we can find a SDR for this family, then we will have a permutation matrix which we can subtract from our magic square of line sum r . We will simply take the permutation matrix with a entries 1 in position (i, j) of the matrix, where j is the representative of A_i in the SDR. Furthermore, what remains will be a matrix with only non-negative integer entries, since we have only taken the entries for the permutation matrix from the SDR. Finally, as we will be removing one from every row and column, the line sum will be $r - 1$.

Let $I \subset [n]$, $|I| = k$. For any k rows of the magic square with line sum r , there are at least k non-zero entries, as there must be at least one in each row. These entries cannot together lie in fewer than k columns. If this were true, then the

The magic square $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ can be expressed as a sum by either

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ or } \\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Figure 1-5: A Magic Cube Expressed as Two Different Sums of Permutation Matrices

sum of the entries would be strictly less than kr , since the columns each sum to r . Therefore, since the sum of these entries is kr , they must lie in at least k columns, and $|A(I)| \geq k$. Hence, by Hall's Marriage Theorem, a SDR exists.

We now have a permutation matrix which, when subtracted from our magic square of line sum r , leaves a magic square of line sum $r - 1$. By induction, we obtain $r - 1$ permutation matrices that sum to the magic square of line sum $r - 1$. Add to them the permutation matrix previously defined, and we have decomposed the magic square of line sum r into r permutation matrices. \square

We note here that this theorem only guarantees that a given magic square can be written as a sum of permutation matrices. We have not proven that a magic square is expressible as a unique sum of permutation matrices. In fact, this is false, assuming $n > 2$. For an example of this, see Figure 1-5.

Now having applied Hall's Theorem, initially a theorem of graph theory, to magic squares, we point out one other interesting connection between graphs and magic squares. We can view an $n \times n$ magic square as a complete bipartite graph, with the rows of the square describing one set of vertices, and the columns the other. Then we label the edge between the i th row vertex and the j th column vertex with the (i, j) entry in the magic square. Having done this, note that the sum of the labels of the edges incident with each vertex is the same. We call

this a magic labeling of the graph. The interesting question now becomes this: What if we consider a graph which is not the complete bipartite graph? For more information on this concept, see Stanley [11].

In the next chapter, we will discuss properties of $H_n(r)$ for fixed r . In particular, we will be interested in establishing recursions for these objects. In this interest, we present a bit of background material in the remainder of this chapter.

Definition 4. A sequence $A(n)$ is *polynomially recursive* (*P-recursive*) if there exist polynomials $P_0(n), P_1(n), \dots, P_k(n)$ such that, for all n ,

$$P_0(n)A(n) + P_1(n)A(n-1) + P_2(n)A(n-2) + \dots + P_k(n)A(n-k) = 0.$$

Definition 5. The *generating function* $G(x)$ for a sequence $A(n)$ is the power series

$$G(x) = \sum_{n=0}^{\infty} A(n)x^n.$$

Definition 6. The *exponential generating function* $E(x)$ for sequence $A(n)$ is the power series

$$E(x) = \sum_{n=0}^{\infty} \frac{A(n)}{n!} x^n.$$

Definition 7. The *doubly exponential generating function* $D(x)$ for sequence $A(n)$ is the power series

$$D(x) = \sum_{n=0}^{\infty} \frac{A(n)}{n!^2} x^n.$$

Definition 8. A power series $A(x)$ is said to be *differentiably finite* (*D-finite*) if the set of derivatives of $A(x)$ spans a finite dimensional vector space over the field of rational functions of x .

The following theorems will be necessary for the results of this dissertation.

Theorem 4. The product of any two P-recursive sequences is P-recursive.

Theorem 5. The product of any two D-finite power series is D-finite.

Theorem 6. *A sequence $A(n)$ is P -recursive if and only if its generating function, $A(x)$, is D -finite.*

In the preceding theorem, we can replace "generating function" with "exponential generating function" or even "doubly exponential generating function."

This follows from the trivial fact that the sequences $\frac{1}{n!}$ and $\frac{1}{n!^2}$ are P -recursive, and therefore, the result of their multiplication by $A(n)$ will remain P -recursive.

For the first appearance of polynomially recursive sequences and differentially finite generating functions, see Stanley [12]. In that paper, in fact, Stanley poses the question: Are the $H_n(3)$ P -recursive? For more examples of P -recursive sequences and differentially finite power series, and the proofs of the Theorems 4, 5, and 6, see Stanley [15].

CHAPTER 2 MAGIC SQUARES

The new results in this dissertation, however, pertain to $H_n(r)$ with fixed r .

Magic squares of the type we have defined seem to have first been considered in 1966. For some initial results, some of which we have stated in Chapter 1 and some of which follow, see [1]. The generating function for the case $r = 2$ was found in that paper to be $H(x) = \frac{e^{x/2}}{\sqrt{1-x}}$. Through the use of this generating function, the following recurrence was found for $H_n(2)$:

$$H_n(2) = n^2 H_{n-1}(2) - (n-1) \binom{n}{2} H_{n-2}(2) \quad (2.1)$$

It follows from this generating function that

$$H_n(2) = \frac{1}{2^{2n}} \sum_{k=0}^n \binom{n}{k} \frac{n!}{(n-k)!} 2^k (2n-2k)! \quad (2.2)$$

It is possible, using the principle of Inclusion-Exclusion, to give a formula for the number of $n \times n$ magic squares of line sum 2 that do not contain any entries equal to 2, and proceed from there. However, even using that method, complicated manipulations involving double sums of binomial coefficients are needed. The formula is nice enough that the existence of a combinatorial proof was conjectured. Here we give that proof. We define a slightly different object, a double magic square. Then we give a closed formula for double magic squares and show that any magic square corresponds with exactly 2^{2n} of the double magic squares. This is the simplest proof for (2.2) that we know, and the proof is not dependent on generating functions.

Definition 9. A row pair in a $2n \times 2n$ matrix consists of row $(2i-1)$ and row $2i$ for some i with $1 \leq i \leq n$. Column pairs are defined similarly. A double magic square of order n is a $2n \times 2n$ matrix with non-negative integer entries which satisfies the following conditions:

- i) The sum of all entries in any row pair or column pair is 2.
- ii) If a 2 is an entry in a row pair (so that all other entries are zero) then it is in the top row of the row pair.
- iii) There is at most one non-zero integer in any row or column.

Let $D_n(2)$ be the number of double magic squares of order n .

Proposition 1. The number of double magic squares of order n is

$$\sum_{k=0}^n \binom{n}{k} \frac{n!}{(n-k)!} 2^k (2n-2k)! \quad (2.3)$$

Proof. The sum is on the number of 2's appearing in the double magic square. If there are k 2's, we choose the row pairs for them in $\binom{n}{k}$ ways, followed by choosing the column pairs in $\frac{n!}{(n-k)!}$ ways. The columns within pairs may be chosen in 2^k ways. Deleting these row and column pairs with 2's produces a permutation matrix of order $2n-2k$; the number of such matrices is $(2n-2k)!$. \square

Proposition 2. There exists a well-defined surjective map f from the set of double magic squares to the set of magic squares such that for each magic square there are exactly 2^{2n} double magic square preimages.

Proof. We define the map f as follows: In the double magic square, sum each of the two rows in each row pair into one row. Then sum the two columns in each column pair into one column. In each case, this is done by simply adding the corresponding entries in the rows (respectively columns). This we call "folding."

The map f is now easily seen to be well-defined; i.e., each double magic square corresponds, under f , to a single magic square. In essence, f takes a double magic square, and equates it with the magic square who, for each (i, j) , has entry (i, j)

$$\text{Example: } \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Figure 2-1: Folding a Double Magic Square

$$\text{Example: } \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Figure 2-2: Another Example of Folding a Double Magic Square

equal to the sum of the four entries in the double magic square obtained by the intersection of row pair i with column pair j .

To prove f is a surjection, take any magic square with k 2s. Consider any $2n$ by $2n$ square with the following properties:

- a) For each position (i, j) in the magic square with a 2, put a 2 in position $(2i - 1, 2j - 1)$.
- b) The only other non-zero entries in the square are 1, and there is a 1 in sector (i, j) iff the (i, j) entry of the magic square is 1.
- c) There is only one non-zero entry in each row and column.

Now, a square of this type will clearly be a double magic square (in fact, there may be more than one double magic square that satisfy these criteria.) Further, this can always be done, as we can easily satisfy conditions *a* and *b*, and then select the positions for the 1's to satisfy condition *c*, since there can only be two 1s in each row (or column) pair. In each row (respectively column) pair, simply place the first 1 in the top row (respectively left column) and the remaining 1 in the bottom row (respectively right column). Hence f is a surjective map.

All that remains is to show that each magic square has 2^{2n} preimages under f . We will do this by "unfolding" a given magic square, and counting the number of double magic squares which will "fold" to our given magic square under f .

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

Figure 2-3: Possibilities for an Entry of 2 in the Double Magic Square

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Figure 2-4: Possibilities for a 2 Unfolding into 1s

Take a magic square with line sum 2 and i 2s. Unfold it in the following way: Double the columns, so that if column i in the magic square had two 1s, the corresponding column pair in the double magic square gets a 1 in each of it's columns. A 2 can be moved to either column in the new pair, or split into two 1s. Next, double the rows. Again, if row i in the magic square had two 1s, the row pair in the double magic square gets a 1 in each row. A 2 must be moved up to the top row of the pair.

It is clear that this will obtain the double magic squares which will "fold" to our magic square. In how many ways can this be done? If a magic square has j 2s, then the columns without a 2 are unfolded in 2^{n-j} ways, the rows without a 2 in the same. This gives $2^{2(n-j)}$ ways to unfold so far. Finally, each 2 in the magic square can be unfolded in 4 ways, as follows: Either it remains a 2, in 2 ways, one for each column in which it can belong, giving two possibilities for the sector in which it belongs, seen in Figure 2-3, or it is broken into 2 ones, in 2 ways, as either one can be in the top row of the pair, as seen in Figure 2-4. This gives 2^{2j} ways to unfold these, for a total of $2^{2n-2j}2^{2j} = 2^{2n}$ magic squares.

□

This method also gives a very nice proof for the recursion of $H_n(2)$. Let $D_n(2)$ denote the number of double magic squares on n . Then, as seen above, $D_n(2) = 4^n H_n(2)$. However, $D_n(2)$ is nothing but the number of permutations on $2n$, with each permutation counted a number of times equal to 2^z , where z is the

number of (i, j) with the sum of the entries common to row pair i and column pair j equal to 2.

Definition 10. A block in a permutation on $2n$ collectively refers to entries $2i - 1$ and $2i$ for some i , $1 \leq i \leq n$.

Definition 11. A number group refers to the elements $2i - 1$ and $2i$ for some i , $1 \leq i \leq n$.

Definition 12. By an r -block we refer to a block of the permutation for which the largest number of elements it contains from the same number group is r .

View each permutation, in one line notation, as a collection of n ordered blocks. Then this permutation is counted, by $D_n(2)$, a number of times equal to 2^z , where z is the number of 2-blocks in the permutation. This brings us to a corollary. This result was known to Anand et. al., but this is a new proof, highly suggestive of our proof of the P-recursion of $H_n(3)$.

Corollary 1. $H_n(2) = n^2 H_{n-1}(2) + \binom{n}{2} (n-1) H_{n-2}(2)$.

Proof. In any permutation on $[2n]$ counted by $D_n(2)$, locate the element $2n$. For the case in which $2n$ is in a block with $2n - 1$, there are $4n D_{n-1}(2)$ possibilities, as follows: There are n choices for the location of the block, 2 choices for the order of the elements in that block, $D_{n-1}(2)$ ways in which the rest of the permutation is counted. Finally, since $2n$ and $2n - 1$ are together, we count this permutation an extra factor of 2.

For the case where $2n$ and $2n - 1$ are separate, we will need to define a slightly different counting function, $Q_n(2)$. To see the reason for this, let us try to proceed as before: $2n$ is in a block with a number that is not $2n - 1$. There are n blocks in which this pair can be found, 2 possible ways to order the block, and $2(n - 1)$ choices for the number paired with $2n$. Now, in how many ways can the remainder of this permutation be placed, counting as in $D_n(2)$?

The answer is that we can treat $2n - 1$ as the number x that we have paired with $2n$. However, when we place the rest of the permutation, $D_{n-1}(2)$ will count the permutation an extra factor of 2 times if $2n - 1$ is placed in a block with the other element of x 's number group. As such, we define $Q_n(2)$ to be the same as $D_n(2)$, excepting that $Q_n(2)$ has one pair of numbers such that, when paired in a block of the permutation, it will not count the permutation that extra factor of 2. Hence we have $D_n(2) = 4nD_{n-1}(2) + 4n(n-1)Q_{n-1}(2)$. All that remains is to express $Q_n(2)$ in terms of D . The following argument quickly establishes the desired relation: $D_n(2)$ counts the permutations on $[2n]$ the desired number of times, except for the case when the 2 numbers whose pairing Q ignores are together. In that case, $D_n(2)$ counts that permutation twice as many times as it should. But this pair of "unpairables" can be together in precisely $2nD_{n-1}(2)$ ways: $2n$ for the block and ordering, and $D_{n-1}(2)$ for the rest of the permutation. This gives $Q_n(2) = D_n(2) - 2nD_{n-1}(2)$. Now combining the $D_n(2)$ and $Q_n(2)$ equations, and using that $D_n(2) = 4^n H_n(2)$, we arrive at the stated recurrence. \square

2.1 Magic Squares of Order n and Line Sum 3

We move the discussion of magic squares along to the case $r = 3$. We were able to enumerate the $H_n(2)$ by looking at a new object, the double magic square. In a similar fashion, we will define a triple magic square, and use it to find a formula for the $H_n(3)$. This is the biggest original contribution of this dissertation. Until now, $H_n(3)$ was known only for a few finite values of n , and these required computation from a recurrence for the symmetric generating function given by Goulden, Jackson, and Reilly. For their method, see [6]. Our formula agrees with the first 20 values computed by their symmetric generating function, according to Table 2-1.

Following the enumeration, we will use the technique of Corollary 1 to prove that $H_n(3)$ is polynomially recursive. Goulden et. al. proved this, but only through

the use of some high powered arguments relying heavily on the use of a computer.

We will not require a computer here.

We begin with some notation.

Definition 13. *Row (respectively column) triple i refers collectively to the rows (respectively columns) $3i - 2$, $3i - 1$, and $3i$. By row (respectively column) triple sum we refer to the sum of the entries of the rows (respectively columns) of the triple.*

Definition 14. *Sector (i, j) is the intersection of row triple i with column triple j , $0 \leq i \leq n$, and $0 \leq j \leq n$.*

Definition 15. *An r -sector is a sector in which the sum of all the non-zero entries is r .*

Let $P_j(n)$ be the number of permutation matrices of size $3n \times 3n$ with exactly j 2-sectors.

We claim the following equation holds:

$$H_n(3) = \frac{1}{6^{2n}} \sum_{k=0}^n \binom{n}{k} \frac{n!}{(n-k)!} 30^k [(3n-3k)! + \sum_{j=0}^{n-k} (2^j - 1) P_j(n-k)] \quad (2.4)$$

We include the formula for $P_j(n-k)$ for the sake of completeness.

Proposition 3. *For all n, k , with $n \geq k$, we have*

$$P_j(n) = \sum_{m=j}^n (-1)^{m-j} \binom{m}{j} \binom{n}{m}^2 18^m m! \left[\sum_{l=0}^m (-1)^l \binom{n}{l} (3n-2m-l)! \right]. \quad (2.5)$$

Proof. View the permutation matrices enumerated by $P_j(n)$ as permutations in one line notation instead of permutation matrices. Sector (i, j) of the matrix contains a non-zero entry if and only if there is an entry in position $3j - 2$, $3j - 1$, or $3j$, with the entry equal to $3i - 2$, $3i - 1$, or $3i$. Hence, sector (i, j) is a 2-sector if and only if there are exactly 2 such entries in those positions from $3i - 2, 3i - 1, 3i$.

Let $Z_m(n)$ be the number of permutations of length $3n$ with at least m blocks which have 2 elements from the same number group. Then

$$P_j(n) = \sum_{m=j}^n (-1)^{m-j} \binom{m}{j} Z_m(n) \quad (2.6)$$

by the Principle of Inclusion-Exclusion. To count $Z_m(n)$, choose the blocks of the permutation to be 2-blocks in $\binom{n}{m}$ ways, and choose the number groups for the entries, also in $\binom{n}{m}$ ways. Order the number groups according to which permutation block into which 2 of its elements will go in $m!$ ways, and select the number to be left out of each number group (so that we do not form 3-blocks when we wish to be forming 2-blocks) in 3^m ways. Then, order each of these m permutation blocks in $3! = 6$ ways, explaining the 18^m term.

The problem here is that we may still, when laying down the rest of the permutation, fill in some of the 2-blocks with the third element necessary to make them 3-blocks. So we will need to sieve these out, in the following way: Let $Y_l(3n - 2m)$ be the number of permutations in which at least l of the 2-blocks are made into 3-blocks by the permutation on $3n - 2m$. Then the number of permutations in which none of the 2-blocks are completed into 3-blocks is $\sum_{l=0}^m (-1)^l \binom{n-k}{l} (3n - 2m - l)!$, again by the Principle of Inclusion-Exclusion.

□

Definition 16. A triple magic square of order n is a permutation matrix of order $3n$ paired with an index number z . The possible index numbers for a given matrix with i 3-sectors and j 2-sectors are the elements of $[6^{i2^j}]$.

Example 1. The matrix in Figure 2-5 has as acceptable index numbers 1, 2, 3, or 4, as it has two 2-sectors and no 3-sectors.

In the use of the index numbers, the number of possibilities is all that matters. The significance of the number 30 in equation (2.4) is that $30 = (3!)^2 - 3!$. As will soon be seen, $(3!)^2$ is the number of ways a row and column will be expanded under

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Figure 2-5: A Triple Magic Matrix

the tripling, but the $3!$ ways in which a 3 by 3 matrix can be filled with ones under the restrictions for the triple magic square we wish to keep. This will allow us to reduce the triple magic squares to magic squares of line sum 1, including those that correspond to this kind of 3-sector, with index numbers from $[2^j]$. Any definition of the matrix that allows for exactly 36 possibilities for the 3-sector expansion works as well.

One such definition is as follows: a 3 can be in any position in the first row of the sector in 3 ways. The other 27 can be garnered by requiring the sector to consist of a 2 and a 1, with the 1 in an equal or lower column, and an equal or lower row, than the 2. This definition, while not requiring index numbers, can be somewhat confusing, and we wish to emphasize the relationship between magic squares of line sum 3 and $3n \times 3n$ permutation matrices.

In the definition, we think of triple magic matrices with at least one 2-sector as being a representative of a class of triple magic squares, each with the same triple magic matrix, but different index numbers.

Theorem 7. *The number of triple magic squares is $6^{2n}H_n(3)$.*

Proof. The $\frac{1}{6^{2n}}$ at the beginning of the formula may lead one to believe that, as was the case with the double magic squares and $H_n(2)$, there are 6^{2n} triple magic squares for each magic square. This is the case only with the inclusion of the index numbers, however, and we must be much more careful. The difficulty comes, as the reader will soon see, with the rows of the $H_n(3)$ having both a 2 and a 1.

Our general method is to try and reduce the triple magic square into a triple magic matrix that is a permutation matrix. The first step will be to count the number of ways we could have 3-sectors in a triple magic square, and then reduce to a smaller triple magic square with no 3-sectors.

Clearly, if a triple magic square has k 3-sectors, we can choose their row triples in $\binom{n}{k}$ ways, followed by their column triples in $\frac{n!}{(n-k)!}$ ways. For each 3-sector, we know that there are $6^k 2^j$ triple magic squares (j remaining the number of 2-sectors) representing this matrix, each with a different index number. We would like to reduce these triple magic squares to smaller squares, and at the same time, remove some of these index numbers. By multiplying the number of triple magic squares with k 3-sectors by 30^k , we have accounted for 5^k of these possibilities, times the 6^k ways in which the k 3-sectors can be written as 3×3 permutation matrices.

Now, deleting these row and column triples, what remains is a triple magic square on $n - k$ with no 3-sectors, excepting 3-sectors composed entirely of 1s. Since each 2-sector has been decomposed into 1s as well, what is left is a $3(n - k)$ by $3(n - k)$ matrix, composed entirely of 1s, with one 1 in each row and column. This is nothing but a permutation matrix, and the number of these is $(3(n - k))!$. In addition, we have now reduced to triple magic squares with effectively no 3-sectors, and so these have much smaller index numbers, namely the elements of $[2^j]$. Hence, the index numbers are dependent only on the number of 2-sectors.

Next we discuss the significance of the second of the summations in our formula. As $P_j(n - k)$ is the number of permutation matrices with exactly j 2-sectors, we count each of these $2^j - 1$ more times, as they have already been counted once in the $(3(n - k))!$.

This gives the desired result for the number of triple magic squares.

□

What is next on our agenda is to establish the relationship between magic squares on n and triple magic squares on n .

Proposition 4. *There are exactly 6^{2n} triple magic squares with line triple sum 3 for every magic square with line sum 3.*

Proof. We expand a magic square into a triple magic square in 2 stages: First, we triple the rows, and then the columns. We do this in such a way that these two steps are independent. That is, we determine the row for every entry in the triple magic square during the first step, and then, in the second step, we determine the column, independent of what row in which each entry lies.

We triple row i in the magic square, so that it now corresponds with row triple i in the triple magic square. The easiest case is a row consisting of only ones. If row i was a row containing three 1s, then the row triple can be any of the $3!$ row triples made by choosing one of the 1s for the top row, one for the middle, and the remaining 1 going to the last row. A column consisting of three 1s is expanded in the same way, with the same number of possibilities.

A row containing a 2 and a 1 is expanded by splitting the 2 into a pair of 1s, and stacking them in the same column. There are 3 choices for which of the two rows are filled by 1s from the 2, and the final 1 goes into the remaining row. Now, taking the column with the two 1s we have just constructed, we can triple it in six ways, as there are now three 1s in this column, and hence the 1-column is tripled as before. To summarize, for every 2, in the tripling of it's row and column, we have 18 total possibilities. Now, for every one of these, we count each triple magic square obtained a factor of two times, for a total of 36.

For every 3 in the magic square, we can expand it's row and column 36 times, 30 as described in condition iv and 6 if it is expanded into all 1s.

Now, as each row and column effectively contributes 6 possibilities in the tripling, we have expanded each magic square into 6^{2n} triple magic squares. Further, there can be no overlap in the classes of triple magic squares corresponding to a pair of magic squares. The reason for this is that we can always tell to which class a triple magic square belongs, for if we sum the entries of each sector, we will obtain the corresponding magic square. \square

We include here the value of $H_n(3)$ for the first 20 values of n , computed through the use of equations (2.4) and (2.5).

n	$H_n(3)$
1	1
2	4
3	55
4	2008
5	153040
6	20933840
7	4662857360
8	1579060246400
9	772200774683520
10	523853880779443200
11	477360556805016931200
12	569060910292172349004800
13	868071731152923490921728000
14	1663043727673392444887284377600
15	3937477620391471128913917360384000
16	11361490405233269088852351904641024000
17	39467705844323372236824674031640375296000
18	163283938767642016139942363441093164154880000
19	796824180968200160537736864316564823259205632000
20	4547339119236385743822415218802509535838329405440000

Let $H_3(x)$ be the doubly exponential generating function for $H_n(3)$. We have now established the following:

$$H_3(x) = \sum_{n=0}^{\infty} \left[\frac{1}{6^{2n}} \sum_{k=0}^n \binom{n}{k} \frac{n!}{(n-k)!} 30^k [(3n-3k)! + \sum_{j=1}^{n-k} (2^j - 1) P_j(n-k)] \right] x^n \quad (2.7)$$

with $P_j(n - k)$ as found above in Proposition 3.

In the following theorem, we will use heavily the notions of P -recursive sequences, Differentiably finite power series, doubly exponential generating functions, and some standard results related to them discussed in Chapter 1.

Let $H_3(x)$ be the doubly exponential generating function of $H_n(3)$.

Theorem 8. *The doubly exponential generating function $H_3(x)$ is D -finite, and hence $H_3(n)$ is polynomially recursive.*

The proof of this theorem will encompass the rest of this chapter.

Firstly, note that we can ignore the $\frac{1}{6^{2n}}$ that begins the formula, since the finite product of P -recursive sequences is P -recursive, and clearly $\frac{1}{6^{2n}}$ is P -recursive.

Now, the rest of the formula can be written as the following product:

$$\left[\sum_{a=0}^{\infty} \frac{1}{a!} (30x)^a \right] \left[\sum_{b=0}^{\infty} \frac{1}{(b!)^2} ((3b)!) + \sum_{j=1}^b (2^j - 1) P_j(b) x^b \right] \quad (2.8)$$

In the above, the coefficient of x^n in the product will correspond with taking $a = k$ and $b = n - k$, and letting k range from 0 to n . Clearly the first of the two terms in the above product is D -finite, since its coefficients are nothing but $\frac{30^a}{a!}$, which is trivially P -recursive. It remains to establish the D -finiteness of the second term. This is the sum of two generating functions, the first of which is also trivially D -finite. Finally, we arrive at $\sum_{b=0}^{\infty} \left[\sum_{j=1}^b (2^j - 1) P_j(b) \right] x^b$, and ask whether this generating function is D -finite, or equivalently, whether it's coefficients are P -recursive. This is exactly the content of the next theorem, whose proof was suggested by the proof of Corollary 1. First we will need a few definitions, similar to those we used to prove the P -recursivity of $H_n(2)$.

Definition 17. *A block in a permutation on $3n$ collectively refers to entries $3i - 2$, $3i - 1$, and $3i$ for some i , $1 \leq i \leq n$.*

Definition 18. *A number group refers to the elements $3i - 2$, $3i - 1$, and $3i$ for some i , $1 \leq i \leq n$.*

Definition 19. By an r -block we refer to a block of the permutation for which the largest number of elements it contains from the same number group is r .

These definitions enable us to handle the enumeration of these permutations with repetition, which corresponds to the index numbers of the triple magic square. That is, since a triple magic matrix corresponds to 2^j triple magic squares, we can simply find the number of 2-blocks in a permutation, and count that permutation 2^j times. This is precisely what we will do in the next section.

2.2 Mind Your P s, Q s, \dots , and T s.

Theorem 9. The sequence $\sum_{j=0}^n (2^n - 1)P_j(n)$ is P -recursive.

Proof. Let $P(n)$ denote the number of permutations on $3n$, with each permutation counted 2^j times, where j is the number of 2-blocks. Locate the element $3n$ in a permutation. Now $3n$ is in a block with both of the other elements from the number group containing $3n$ in $6nP(n-1)$ ways, as follows: There are n choices for the block containing these 3 elements, 6 choices for the number of ways in which we can order them, and $P(n-1)$ ways in which the rest of the permutation can appear, counted properly with respect to the number of 2-blocks.

On the other hand, if $3n$ is in a block with only one other member of his number group, then we can choose the third element, say z , for this block in $3(n-1)$ ways, the block for these in n ways, and order them in the block in 6 ways. Also, we need to count each of these permutations twice, since $3n$ has appeared in a block with another element of his number group. But what about the rest of the permutation? What remains is $3(n-1)$ entries, along with every element in $[3n]$ save $3n$, one of $3n-1$ and $3n-2$, and one other entry from $[3(n-1)]$.

Suppose that $3n-2$ is the remaining element. We cannot simply identify $3n-2$ with the missing element, and say that there are $P(n-1)$ ways of completing the permutation, because $3n-2$ may be in a block with one (or both) of the other elements from the number group belonging to z . If this is the case, $P(n-1)$ will

not count this permutation properly with respect to the counting of 2-blocks we require.

Example 2. *The 9-permutation 2 4 3 7 6 1 8 5 9 should be counted four times by $P(n)$, since 2 and 3 are together in the first block, and 8 and 9 are together in the third block. If we take the 859 block out, and consider 7 to be replaced by 5, we arrive at 243561. $P(2)$ will count this permutation 4 times. We want it to be counted twice, as the 5 (really 7) should not make the permutation be counted an extra factor of 2 times.*

So we define a new function, $Q(n)$. Now $Q(n)$ counts permutations the same way as $P(n)$, except that in Q , one element will not combine with another in his number group to form a 2-block or 3-block. This element, when forming a 2-block, will not force this double counting. Furthermore, if this element is in a block with both of the other elements belonging to the same number group, $Q(n)$ will count this block as a 2-block, instead of a 3-block.

With this definition, we see that when $3n$ is in a block with one other element from his number group, we have $36n(n-1)Q(n-1)$ permutations of this type. Now we need to show that $Q(n)$ can somehow be expressed in terms of $P(n)$, if we hope to use $Q(n)$ to show the P-recursivity of the $P(n)$.

First, we require a definition that will be used heavily in the remainder of this chapter, followed by a few examples to illustrate this concept.

Definition 20. *An unpairable is an element of the permutation that, when placed in a block with other elements from his number group, is ignored in the determination of whether that block is a 2-block or 3-block.*

Example 3. *The 5 in our last example is an unpairable, after the removal of the block and number group containing 9.*

Example 4. The 12-permutation 3 1 2 6 8 4 5 12 10 9 11 7 should be counted a total of 8 times by $P(4)$. Removal of the third block, we have remaining 3 1 2 6 8 4 9 11 7. We now wish to remove the number group 10, 11, 12 from the permutation entirely. As the 11 is the only element yet remaining from that number group, we replace 11 by the 5 we removed with the third block, and arrive at 3 1 2 6 8 4 9 5 7. The 5 is now an unpairable, but since it is located in a block without 4 or 6, $P(n)$ counts this permutation properly, a total of 4 times. This is why we have included an extra factor of 2 in the coefficient of $Q(n)$, above.

Example 5. The 9-permutation 3 8 2 1 7 9 4 6 5 should be counted 4 times by $P(n)$. In the procedure we are following, the second block of the permutation is removed, along with the number group 7, 8, 9. The 8 then is replaced by 1, leaving the 6-permutation 3 1 2 4 6 5, with 1 an unpairable. We are already double counting this permutation, since the original second block was a 2-block. Now, the remaining permutation would be counted by $P(n)$ only once, since both blocks are 3-blocks. However, with 1 an unpairable, $Q(n)$ considers the first block a 2-block, as it ignores the element 1, and double counts the permutation. Hence, with the unpairable, the permutation is counted the proper number of times the terms we have so far found for $P(n)$.

Lemma 2. For all positive integers $n \geq 2$ we have

$$Q(n) = P(n) + 6nP(n-1) - 36n(n-1)Q(n-1) \quad (2.9)$$

Proof. The one element remaining from the number group belonging to $3n$ is an unpairable. If the unpairable is in a block with no other element of his number group, then $P(n)$ will count the permutation the proper number of times. (That is, $Q(n)$ will agree with $P(n)$ on the number of such permutations).

On the other hand, if the unpairable is in a block with both other elements of his number group, then $Q(n)$ will only count the permutation half as many times

as $P(n)$, since $P(n)$ will view the block as a 3-block and count the permutation accordingly, while $Q(n)$ will see this block as a 2-block, and double count the permutation for this particular block. So the $P(n)$ term above will count the permutation half as many times as necessary, and these three elements can be together in the same block precisely $6nP(n-1)$ times, and hence adding this to $P(n)$ corrects the number of this type of permutation.

If the unpairable is in a block with exactly one other element of it's number group, $Q(n)$ will view this particular block as a 1-block, but $P(n)$ sees it as a 2-block, and counts it twice as much as $Q(n)$. This can happen in $36n(n-1)Q(n-1)$ ways, as follows: There are 2 choices for the other element from the number group of the unpairable, $3(n-1)$ choices for the third element for the block, $6n$ choices for the block and the ordering of the elements of the block, and then $Q(n-1)$ choices for the rest of the permutation. The remainder of the permutation has a new unpairable, created by the removal of the first unpairable, the other element from that number group, and the third element that went into the block. Now the one element remaining from the first unpairable's number group must take the place of the other element in the block we are removing. This term we subtract from the two previous, to correct for $P(n)$ over counting the permutations in this case.

Hence, we have the result. □

As we now see, $Q(n) + 36n(n-1)Q(n-1) = P(n) + 6nP(n-1)$, and we still have hope that, after we get a complete formula for $P(n)$, we may be able, through the use of the lemma, establish a recurrence for $P(n)$.

We have dealt with the cases of permutations in which $3n$ is in a block with one or both of the other elements of it's number group. We now turn our attention to the final case: That of $3n$ in a block with no other member of it's number group.

Here we have two subcases, whether $3n$ is in a block with a pair of elements from the same number group, or not.

If $3n$ is in a block with a pair of elements from the same number group, then we have $(n-1)\binom{3}{2}$ choices for the pair of elements, and again $6n$ choices for the block and the order of the elements in said block. Setting aside this block, what remains is every element but $3n$ and this pair. If we identify $3n-2$ and $3n-1$ with this pair, we have the set $[3(n-1)]$ of elements, with one unpairable. The one element left from the original number group of the pair which were in a block with $3n$ is now in the same number group as $3n-2$ and $3n-1$. So we consider that element the unpairable, and have $Q(n-1)$ ways to complete the permutation. However, we must remember that, in the original permutation, there was a 2-block that we have removed, and so we need to double count these permutations, for a grand total of $2(n-1)\binom{3}{2}6nQ(n-1)$.

If $3n$ is in a block with two elements from different number groups, things turn a little more complicated. Now if we remove this block, and identify $3n-2$ and $3n-1$ with the elements paired with $3n$, we will have two number groups which each have an element that will not properly combine with the others in their number groups. Further, if one puts these in the same block, since they are really $3n-1$ and $3n-2$, they should make that block a 2-block! Unfortunately, we need to define a new function here, $R(n)$, that does just this, and try to relate it with $P(n)$ and $Q(n)$.

Definition 21. *The function $R(n)$ counts permutations on $[3n]$ in the same way as $P(n)$, except there are 2 unpairables, each in a different number group, which, when placed in the same block, make that block a 2-block.*

Example 6. *Suppose the 12-permutation $3\ 6\ 4\ 9\ 12\ 10\ 7\ 8\ 1\ 2\ 11\ 5$ has, as unpairable elements, 4 and 10. Then $P(n)$ counts this permutation eight times.*

$R(n)$ counts it only twice. $P(n)$ sees blocks 1, 2, and 3 of the permutation as 2-blocks. $R(n)$ agrees that the second block is a 2-block, but sees both blocks 1 and 3 as 1-blocks, since 4 and 10, the unpairables, were what made those blocks into 2-blocks.

Example 7. Suppose the 9-permutation 3 6 8 2 4 1 9 7 5 has 3 and 6 for unpairables. $P(n)$ would count this permutation four times. $R(n)$, on the other hand, counts the permutation eight times. Both functions agree that blocks two and three are 2-blocks. However, $R(n)$ also sees the first block, with the two unpairables, as a 2-block.

Example 8. Suppose the 9-permutation 2 4 1 5 9 7 3 6 8 has 2 and 4 for unpairables. $P(n)$ and $R(n)$ both count this permutation four times, but for different reasons. Both functions agree that the second block is a 2-block. They also agree that the first block is a 2-block: $P(n)$ for there being a 1 and 2 in the block, but $R(n)$ for the two unpairables being in the block together, but ignoring that 1 and 2 are together.

We now return to the proof of the theorem, using the new function to complete a formula for $P(n)$. Now, with $3n$ in a block with two elements of distinct number groups, this contributes $54n\binom{n-1}{2}R(n-1)$ to $P(n)$, since we can choose the two elements for the block with $3n$ in $9\binom{n-1}{2}$ ways, 9 for the actual elements and $\binom{n-1}{2}$ for the two number groups in which these elements belong. Again, we have $6n$ choices for the block and the order of the elements in the block, and $R(n-1)$ for the remainder of the permutation.

In total, we now have

$$P(n) = 6n \left(P(n-1) + 36n(n-1)Q(n-1) + 54n\binom{n-1}{2}R(n-1) \right). \quad (2.10)$$

Now we establish the relationship between $R(n)$, $Q(n)$, and $P(n)$. This last case, where the $R(n)$ term arose, is the most challenging, as we will need to

define two more functions, just to write a formula for $R(n)$. Fortunately, these two functions have a much easier relation to $P(n)$, $Q(n)$, and $R(n)$, and are independent of one another. Then, between these five expressions, we will be able to show a recursion for $P(n)$. As a side note, this will also prove, after showing P-recursivity of $P(n)$, the P-recursivity of $Q(n)$, $R(n)$, and the two functions whose definitions follow.

Definition 22. *The function $S(n)$ counts permutations on $3n$ with two unpairable elements in different number groups, and otherwise counts permutations in the same manner as $P(n)$.*

Definition 23. *The function $T(n)$ counts permutations on $3n$ with two unpairables belonging to the same number group, and otherwise counts permutations in the same manner as $P(n)$.*

To speak more informally, $S(n)$ counts permutations as many times as $R(n)$, except that the two unpairables from $S(n)$ will not combine to form 2-blocks. So $S(n)$ is a simpler function than $P(n)$. $T(n)$ having two unpairables from the same number group just means that there is a number group in which none of the elements will combine to form 2-blocks.

Now, we return our attention to $R(n)$ with the following lemma:

Lemma 3. *For all positive integers $n \geq 1$, we have*

$$R(n) = Q(n) + 6n(3Q(n-1) - 6(n-2)S(n-1) - 4T(n-1) + 3(n-2)R(n-1)). \quad (2.11)$$

Proof. We begin the expression for $R(n)$ with a $Q(n)$ which counts the permutations so that one of the unpairables is already counted properly when placed in a block with elements from his own number group. It remains to deal with the other

unpairable, and the case in which the two unpairables are together. The first part of this argument, therefore, will be similar to the proof of Lemma 1.

If the unaccounted for unpairable is in a block with both other elements from his number group, then $Q(n)$ has counted the permutation only half as many times as necessary, as it considers the block to be a 3-block when, according to $R(n)$, it is a 2-block. This can happen in $6nQ(n-1)$ ways, $6n$ again for the block and order of elements in the block, and $Q(n-1)$ for the rest of the permutation to count properly with the one remaining unpairable.

If the unpairable is in a block with one other element from his number group, the permutation has been double counted, since $R(n)$ will consider this a 1-block, but $Q(n)$ will consider it a 2-block. We need to be careful, however, because the third element in the block may be the other unpairable. If this is the case, then $Q(n)$ has already counted this permutation the proper number of times, as it considered the unpairable to pair with the other element from its number group, but did not pair the two unpairables, which $R(n)$ does. So we can take the third element of this number group to not be the other unpairable.

Even so, we still must be careful, for the third element in this block may be one of the two elements belonging to the other unpairable's number group. In this case, there are 2 choices for this element, $6n$ choices for the block and order of the elements in the block, and $T(n-1)$ choices for the remainder of the permutation. Also, there are two choices for the element from the number group of the unpairable. The $T(n-1)$ arises because we remove from the permutation the one of the unpairables and one other element of his number group, leaving one element from that group left. That element takes the place of the element we took from the other unpairable's number group, leaving two unpairables in the same number group in the resulting permutation on $3(n-1)$. Hence, we need to subtract $24nT(n-1)$.

Now, for the end of this particular case, we take the third element in the block to be from a different number group than both the unpairables. We have $n - 2$ choices for this number group, and 3 choices for the element therein. There are again 2 choices for the element we are taking from the unpairables number group. Now, removing this, we have created another unpairable, the one remaining element from the unpairables number group. We want it to take the place of the element we chose to complete the block, which was in a different number group than both those that originally contained unpairables. Hence, we have a permutation on $3(n - 1)$ remaining, with two unpairables in different number groups. Further, these two unpairables will not combine to form a 2-block. As defined, this can happen in $S(n - 1)$ ways. So for this case, we need to subtract $36n(n - 2)S(n - 1)$ from $Q(n)$, so that now permutations of this type are counted properly by $R(n)$.

If the unpairable is by itself in a block with no other members of his number group, nor the other unpairable, the permutation is counted the proper number of times by $Q(n)$. All that remains is to correct for $Q(n)$ when the two unpairables are together in a block. Now, if the third element is from a number group of one of the unpairables, there are two different cases to examine. If the third element is from the same number group as the unpairable that $Q(n)$ has already considered, then the permutation has not been counted properly. However, if the third element of the block belongs to the number group corresponding with the other unpairable, the block has been considered properly as a 2-block, since $Q(n)$ will count it as a 2-block for these two, while $R(n)$ considers the block a 2-block because of the unpairables residing therein.

For the case in which the block contains both unpairables and an element from the group belonging to the first unpairable, $Q(n)$ has counted these permutations half as many times as necessary, since it considers this block a 1-block, while $R(n)$

considers this block a 2-block. This can happen in $12nQ(n-1)$ ways: 2 for the third element of the block from the unpairable's number group, $6n$ for the block and order of elements in the block, and $Q(n-1)$ since the remaining permutation has only one unpairable. We have removed the original 2 unpairables, but now the one remaining element from the number group from which we have taken two elements takes the place of the other unpairable.

Finally, if the two unpairables are in the same block, with the third element from a different number group, $Q(n)$ has counted these permutations half as many times as $R(n)$, since $R(n)$ considers a block with both unpairables to be a 2-block. There are $3(n-2)$ choices for this third element. What remains after we remove this block is a permutation on $3(n-1)$, with three number groups with only 2 elements. If we take one of these groups and use its elements to complete the other two number groups, we will have two unpairables in different number groups which would combine with each other in a block to make a 2-block. In other words, the remainder of the permutation has $R(n-1)$ possibilities.

Adding (and subtracting) together these terms we have found brings us to the desired result. □

Corollary 2. *For all positive integers $n \geq 2$, we have*

$$S(n) = R(n) - 6n(4Q(n-1) + 3(n-2)R(n-1)). \quad (2.12)$$

Proof. $S(n)$ has a very similar structure to $R(n)$. In fact, the only difference is that $S(n)$ does not consider the two unpairables together in a block to make that block a 2-block. Therefore, we merely take $R(n)$ and subtract out the two terms, $12nQ(n-1)$ and $18n(n-2)R(n-1)$ that we added to account for this, save for one case. In the proof of the formula for $R(n)$, when the two unpairables were in the same block with an element from the number group belonging to the unpairable not known to $Q(n)$, $Q(n)$ counted that as a 2-block already, so we needed no

correction factor. Now, here in $S(n)$, this is not a 2-block at all, and hence $R(n)$ counts these permutations twice as much as necessary. Hence we subtract another $12nQ(n-1)$, since there are 2 choices for the third element, $6n$ ways for the block and ordering, and $Q(n-1)$ ways for the remainder of the permutation, since it will only have as an unpairable the final element from the number group belonging to the original unpairable not known to $Q(n)$. \square

$T(n)$, mercifully, is a simpler function. We only have to account for one of the number groups not having its elements form a 2-block if placed in the same block.

Proposition 5. *For all positive integers $n \geq 2$, we have*

$$T(n) = P(n) - 54n(n-1)Q(n-1).$$

Proof. Start with $P(n)$. If all three elements from this number group are in the same block, $P(n)$ considers the block to be a 3-block, while $T(n)$ considers the block a 1-block. Each function counts this permutation the same number of times. If two of these elements are together, according to $P(n)$ there is a 2-block, but $T(n)$ thinks it is a 1-block. Hence, $P(n)$ has double counted these permutations, and so we need to subtract $18\binom{3}{2}n(n-1)Q(n-1)$: $6n$ for the choice of block and order in the block, $\binom{3}{2}$ for the two elements from the number group, $3(n-1)$ choices for the third element of the block, and $Q(n-1)$ for the remaining permutation which has the remaining element of our nonfunctional number block as an unpairable. Finally, if all three elements of the number group are in different blocks, $P(n)$ and $T(n)$ agree on the number of this type of permutation. \square

At this point, it will be useful to summarize the formulas for the five functions we have previously obtained.

$$P(n) = 6n \left(P(n-1) + 36n(n-1)Q(n-1) + 54n \binom{n-1}{2} R(n-1) \right). \quad (2.13)$$

$$Q(n) = P(n) + 6nP(n-1) - 36n(n-1)Q(n-1). \quad (2.14)$$

$$R(n) = Q(n) + 6n(3Q(n-1) - 6(n-2)S(n-1) - 4T(n-1) + 3(n-2)R(n-1)). \quad (2.15)$$

$$S(n) = R(n) - 6n(4Q(n-1) + 3(n-2)R(n-1)). \quad (2.16)$$

$$T(n) = P(n) - 54n(n-1)Q(n-1). \quad (2.17)$$

Now we simplify these five equations through some simple algebraic manipulation to arrive at an expression involving only the function P . Note that, intuitively, we can see that $P(n)$ is P -recursive from the formulas above. The reasoning is as follows: If we can get an expression with only P s and Q s, we will be done, as we can use 2.14 to reduce that expression to one involving only P . In some sense, then, $T(n)$ is simple to account for, as it already involves only P and Q . The procedure, then, will be to remove the R term from 2.13 by using the fact that, in 2.15,

$$R(n) + 18n(n-2)R(n-1) = Q(n) + 6n(3Q(n-1) - 6(n-2)S(n-1) - 4T(n-1)). \quad (2.18)$$

Then we will have an equation that has only P , Q , S , and T . But then, using 2.18 in equation 2.16, we can write S as a function of Q , S , and T . Then, using this, we can remove S from the P formula by recursion, leaving only P , Q , and T . Finally, using 2.14, we will reduce to an equation involving only $P(n)$, $P(n-1)$, $P(n-2)$, $P(n-3)$, and $P(n-4)$ with polynomial coefficients.

After this, one arrives at the following equation, which concludes the proof of Theorem 9.

$$\begin{aligned}
& (n-2)P(n) + 6n(n^2-10)P(n-1) - 972n(n-1)^2(n^2-6n+10)P(n-2) \\
& + 3888n(n-1)^2(n-2)^2[324n^5-6804n^4+41148n^3-105948n^2+121828n-50555]P(n-3) \\
& + 69984n(n-1)^2(n-2)^2(n-3)^2(2n-5)P(n-4) = 0. \tag{2.19}
\end{aligned}$$

□

CHAPTER 3 MAGIC CUBES

In this chapter we will discuss a 3-dimensional analogue of the magic square, the magic cube. Recall from chapter 1 the distribution problem of the fertilizer. Now suppose that we fertilize our plot of land once a week. Place the further condition that, over n weeks, each subsquare of our plot of land receives the same amount of fertilizer.

The solution to this problem will be a bit more difficult. Now, we have n magic squares, one for each of the n weeks, in which we have fertilized our plot of land. However, for each of the n^2 entries (i, j) in the magic squares, when we sum through the magic squares, we must again obtain the sum r . As magic squares were the solution to the original problem, magic cubes will be the solution to this problem.

Definition 24. *A magic cube of order n and line sum r is a 3-dimensional $n \times n \times n$ array of non-negative integers such that each $k \times n \times n$ level is a magic square of line sum r , as well as the $n \times k \times n$ and $n \times n \times k$ levels, where k is fixed.*

Example 9. *Figure 3-1 shows the four levels of a magic cube from top to bottom, i.e. the first square is level $4 \times [4] \times [4]$ of the cube, and the last square is level $1 \times [4] \times [4]$ of the cube.*

$$\begin{pmatrix} 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 1 \\ 0 & 3 & 1 & 0 \\ 1 & 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 2 & 0 & 1 & 1 \\ 1 & 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 3 & 1 & 0 \\ 1 & 0 & 0 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Figure 3-1: A Magic Cube of Order 4 and Line Sum 4

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Figure 3-2: A Magic Cube of Line Sum 1

$$\begin{pmatrix} 4 & 2 & 1 & 3 \\ 2 & 1 & 3 & 4 \\ 1 & 3 & 4 & 2 \\ 3 & 4 & 2 & 1 \end{pmatrix}$$

Figure 3-3: A Latin Square

Let $C_n(r)$ be the number of magic cubes of order n and line sum r . Enumeration of the magic cubes has proven difficult over the years. We can see why this is true simply by examining $C_n(1)$. That is, how many magic cubes are there of order n and line sum 1? At first glance, it may seem that this cannot be very difficult, as each level of the cube is a permutation matrix. However, one quickly realizes the complexity of the subject when one considers the ramifications of these permutation matrices being stacked on top of each other, under the condition that the line sum is also one for a vertical column of the cube.

In this case, we could view the cube as permutations which avoid each other. That is, the cube will correspond with a collection of n permutations with the property that the i th entry is different in each permutation. Equivalently, if we choose the i th entry of each permutation, we will have another permutation of n .

Example 10. *For the magic cube of order 4 and line sum 1 in Figure 3-2 we have the permutations 4213, 2134, 1342, and 3421.*

We can more easily see that the i th entries are permutations if we write the permutations into an $n \times n$ matrix. By convention, let us put the permutation corresponding to the top magic square of the cube on top, and proceed to the bottom. We arrive at Figure 3-3. Note that in every row and column of the matrix, we have a permutation. This object is called a Latin Square. Latin Squares have

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Figure 3-4: An Indecomposable Magic Cube

been studied for many years, and yet not much is known about their enumeration. We have some bounds for the number of them, but no formula has ever been found.

In general, it is true that the magic cubes of order n and line sum 1 are in bijection with the Latin Squares of order n . This gives testament to the complexity of magic cubes, as even when $r = 1$, we have something which we cannot enumerate.

To see another reason why magic cubes are much more difficult to enumerate than magic squares, consider one of the main properties of magic squares. As we have seen, magic squares have extensive relationships with permutations, or magic squares of line sum 1. The results of chapter 2 depend mostly on reduction to permutation matrices in one way or another. Recall from chapter 1 that magic squares of line sum r can always be reduced to a sum of permutation matrices. With magic cubes, we cannot generalize this result. That is, it is not true that we can reduce any magic cube with line sum r to a sum of magic cubes with line sum 1, or Latin Squares. One such cube is given in Figure 3-4. This cube, in fact, is only of order 3 and line sum 2, and even with n and r this small, we cannot reduce this cube to a sum of two Latin Squares (i.e. magic cubes of line sum 1.) For more results and discussion on these "irreducible" magic cubes, see Bóna [2], Bóna [5], and Stanley [14].

Note that, in the preceding example, the top level of the cube can only be decomposed into the squares of Figure 3-5 because of the 2 in the upper left hand corner of the square. Any attempted decomposition of the other two levels will not combine with these two squares to make two magic cubes of line sum 1.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Figure 3–5: Decomposed Top Level

We can still use, however, some properties of Latin Squares to garner some information about magic cubes. In this chapter, we try to visualize magic cubes as a kind of generalization of Latin Squares, and prove an initial result in this direction.

By a partially full Latin Square, we mean an $n \times n$ matrix which is only partially full, with no integer appearing more than once in any row or column. Most results on Latin Squares come in the form of conditions we can place on a partially full Latin Square so that we are guaranteed to be able to complete the square. Following these lines of argument, our result for this chapter is one such condition we can place on a partially filled magic cube, so that we are guaranteed to be able to complete the cube.

Our claim is that given a partial magic cube consisting of l completed levels, with the remainder of the cube empty, the partial magic cube can be completed to a magic cube. To accomplish this, we relate the magic cube of line sum r to a different array of numbers which more closely resembles a Latin Square.

3.1 A Magic Rectangle

For each level of a magic cube, begin a new array by the procedure described in the following paragraphs.

In the first row of the top magic square, the line sum is r . Locate the column of each non-zero entry in the first row, and write, in the first row of the new array, the integer corresponding to the column number of each non-zero entry. If a number larger than 1 appears, write the column number the same number of times as the entry itself. Continue with the first row of the magic square which is the next level of the cube, writing the column numbers of its entries with repetition.

$$\begin{pmatrix} 2 & 3 & 1 & 2 & 1 & 3 \\ 2 & 3 & 1 & 3 & 1 & 2 \\ 1 & 1 & 2 & 3 & 2 & 3 \end{pmatrix}$$

Figure 3-6: A Magic Rectangle

Repeat this process through the bottom magic square of the cube. Now for the second row of the array, write the column numbers of each non-zero entry in the second row of each magic square, with repetition as before, from the top magic square to the bottom magic square of the cube. Then repeat, each row of the new array being built from the column numbers of the corresponding row in each level of the magic cube. The array obtained we call a magic rectangle.

Example 11. *For the magic cube of Figure 3-4 we have the magic rectangle of Figure 3-6 (as one of the possibilities).*

This rectangle will have the following properties:

- i) In each row, each integer from 1 to n will appear r times.
- ii) Each integer will appear r times total among columns $1-r, (r+1)-2r, \dots$

We say that the magic rectangle of Figure 3-6 is one of the possibilities for the magic cube pictured in Figure 3-4 because we have placed no condition on the actual order of the entries in the first r columns, nor in columns $r+1$ to $2r$, etc. Hence, we have more than one magic rectangle for any particular magic cube. Next, we will describe a particular type of magic rectangle which we will be using hereafter.

Because of the fact that any magic square can be broken down into a sum of permutation matrices, we can rearrange the numbers in the first r columns in their respective rows to arrive at a magic rectangle with the property that each of the first r columns is a permutation. Likewise, one can also rearrange the numbers in columns $r+1$ to $2r$ in their respective rows, so that the next r columns are also permutations. Continuing in this fashion, we arrive at a magic rectangle which has the property that all of its columns are permutations. Furthermore, this magic

$$\begin{pmatrix} 2 & 3 & 1 & 2 & 1 & 3 \\ 3 & 2 & 3 & 1 & 2 & 1 \\ 1 & 1 & 2 & 3 & 3 & 2 \end{pmatrix}$$

Figure 3-7: Magic Rectangle Reorganized into Columnwise Permutations

rectangle will still correspond with the magic cube with which the process began. Our example could be expressed as in Figure 3-7.

Note that, although we can make these columns permutations, we cannot always partition this array into Latin Squares. This is the reason why a magic cube of order n and line sum r cannot always be expressed as a sum of magic cubes of order n and line sum 1. This is another example of the ways in which magic cubes are much more complicated than magic squares.

The magic cube can be easily rebuilt, merely by noting that the first row of the magic rectangle determines the magic square $[n] \times 1 \times [n]$ magic cube, the second row gives the $[n] \times 2 \times [n]$ magic square, and so forth. Continuing in this fashion, we arrive at the magic cube, using each row in the following way:

The first r columns determine the column number of each entry in the top row of the square, the next r columns for the next highest row, and so on.

Thus, the question of completing a magic cube given the first few complete levels reduces to that of completing a Magic Rectangle for which we know the first few complete rows.

3.2 Marrying People

What follows is similar to the proof of the fact that a Latin Square can be completed given the first few rows of the square. The latter can be proven using Philip Hall's Marriage Theorem. For this application of Hall's Theorem, see [16]. In the language of magic cubes, this proves the desired result for cubes of line sum 1, which are known (and can be seen by the procedure outlined in this dissertation) to be in bijection with Latin Squares. To prove the theorem for higher values of r , we will need to generalize Hall's Marriage Theorem. We use the following theorem

to this purpose, which originally appeared in a book by L. Mirsky. For this proof, along with other generalizations of Hall's Theorem for SDRs, see [10]. For the case $r = 1$, this theorem has Hall's theorem as a consequence. Recall from chapter 1 the definition of a system of distinct representatives.

Recall that by $|A(I)|$ we mean the cardinality of $\bigcup\{A_i : i \in I\}$.

Theorem 10. *Let (A_1, A_2, \dots, A_n) be a family of sets and let r be a positive integer. Then it is possible to partition this family into r subfamilies, each of which possesses an SDR, if and only if*

$$r|A(I)| \geq |I| \quad (3.1)$$

for all $I \subset [n]$.

Proof. Examine the family

$$F = \{A_1 \times [r], A_2 \times [r], \dots, A_n \times [r]\}.$$

By Hall's theorem, this family has an SDR if and only if, for each $I \subset [n]$,

$$|I| \leq \left| \bigcup F \right| = |A(I) \times [r]| = r|A(I)|. \quad (3.2)$$

By hypothesis this condition is satisfied, and therefore our set has an SDR. Denote this SDR by the elements (x_i, k_i) with $x_i \in A_i$, $1 \leq i \leq n$, $k_i \in [r]$. Let

$I_j = \{i : 1 \leq i \leq n, k_i = j\}$ for each j , $1 \leq j \leq r$. Now, (I_1, \dots, I_r) is a partition of $[n]$, and since the collection of (x_i, k_i) is distinct, the subfamily indexed by I_j has as its SDR the elements x_i given by the (x_i, k_i) , $k_i = j$. Thus we have partitioned into subfamilies, each of which has an SDR.

Conversely, suppose that we can partition our family into r subfamilies, each with an SDR. Assign the integers from $[r]$ to the subfamilies, and index the subfamilies by I_i . Then for the subfamily indexed by I_i with SDR $\{x : x \in \text{SDR}(I_i)\}$, write for this subfamily the set of pairs $\{(x_m, i), 1 \leq m \leq |I_i|\}$. Then clearly the set containing these pairs from each subfamily is an SDR for

the family $(A_1 \times [r], \dots, A_n \times [r])$. By Hall's theorem, $|A(I) \times [r]| \geq |I|$. But $|A(I) \times [r]| = |A(I)|r$, and the condition of the theorem is satisfied. \square

First we will need to define our family of sets. For any partially completed Magic Rectangle, in the sense that we have the first few complete rows, let A_i denote the set of numbers from $[n]$ that have not yet appeared in column i . All that remains is to find a partition of these rn sets into r subfamilies, each of which will necessarily be of size n , such that there is an SDR for each subfamily. If we can do this, then we will have a next possible row for this partial Magic Rectangle. One simply places the element of n that is the representative for set A_i as the entry for that column in a new row. This row will have rn entries, and the columns will still be partial permutations, as the new entries are numbers that had not previously appeared in their columns. Further, each number must appear exactly r times in the row. Hence, we will have extended to a $k \times rn$ partial magic rectangle to a $(k + 1) \times rn$ magic rectangle. Then, by induction, we can complete the rectangle, and hence, the magic cube.

Theorem 11. *Any partial $k \times n$ magic rectangle can be extended to a $(k + 1) \times n$ magic rectangle.*

Proof. Let $j \in [n]$. Then j has been an entry r times in each row of the partial magic rectangle. Hence j has occurred kr times in the partial magic rectangle. Then j is an element in $(n - k)r$ of the A_i . Now, any collection of p of the A_i has $p(n - k)$ total elements, and therefore at least $\lceil \frac{p}{r} \rceil$ distinct elements. For $I \subseteq [n]$, $|I| = p$, $r|A(I)| \geq p$, so that this collection A satisfies the condition of Theorem 6.

By Theorem 6, a partition of this family with the desired properties exists. Hence, we have the result. \square

CHAPTER 4 SUMMARY AND CONCLUSIONS

There is much to study still on magic squares. It was in Stanley's paper "Differentiably Finite Power Series" that he first questioned the P -recursivity of magic squares. In fact, this was one of the first objects for which the question was asked. At that time, P -recursion of $H_n(2)$ was already known, but for no other r was there an answer. Finally, in 1983, came an answer for the case of $r = 3$. However, this proof used some very high-powered tools, such as Hammond series of symmetric functions, and, even worse, a computer program called VAXIMA was required to perform necessary computations in the proof.

What we have sought in this dissertation is a more bijective, combinatorial procedure for deciding the P -recursivity of magic squares. With the use of Double Magic Squares, we have nice proofs not only for the P -recursivity, but also for the number $H_n(2)$. We have been able to generalize these results for $H_n(3)$, without the need for any particularly advanced tools or programs. Further, using the technique, we are now able to enumerate the $H_n(3)$.

The obvious next question to be answered is this: What can be said about $H_n(4)$? It is easy enough to see that what we will need is the P -recursivity of something similar to that of the function of Theorem 9. That is, we need to count the number of permutations of size $4n$ in a special way, as we did for the permutations of size $3n$. Now we need to look at each block of the permutation, and find the number of blocks with two or three elements from the same number group. Further, since a block could have two elements from two different number groups, we would need to count the number of such blocks. Then, if there are k blocks with two elements from two different number groups, l blocks with one such

pair, and m blocks with three elements from the same number group, we will need to count the permutation $4^k 2^l 6^m$ times.

For the purpose of the following discussion, two elements are said to "pair" if, neither of the two are unpairable with respect to the other.

As before, let $P(n)$ be the number of permutations of size $4n$ counted in the aforementioned fashion. As before, we can easily begin to apply the same ideas from Chapter 2 to this number. That is, we can say that there are $24nP(n-1)$ possibilities for the number of permutations with $4n$ in a block with all the elements of that number group, $96n(4n-4)Q(n-1)$ possibilities for a permutation with $4n$ in a block with two of the three other elements from that number group; here $Q(n)$ is the same as $P(n)$ save for the existence of an unpairable, etc.

The problem with this comes in when we examine the case of permutations in which $4n$ is in a block with one other element of the same number group, along with two elements of a different number group. We then arrive at $432n(n-1)R(n-1)$ possibilities, where $R(n)$ is identical to $P(n)$, except $R(n)$ treats one number group differently. In $R(n)$, there is one number group in which two elements pair with each other, as do the remaining two elements, but between these two "subgroups," pairing will not occur. For example, the number group 1, 2, 3, 4 where elements 1 and 2 will pair, 3 and 4 will pair, but 1 and 3 will not, etc.

As it turns out, using this will result in needing an expression also for the same condition when one of the two will pair, but the other not. In addition, we will need to account for the case of 3 different number groups, each with an unpairable, but with the unpairables combining together to form 2-blocks. Further, we will have to handle cases in which some of the unpairables will pair together, while others will not. In short, we quickly run low on letters of the alphabet to track the different necessary functions, and as of yet, we have not been able to find a way to eliminate a significant portion. The problem becomes very cluttered,

especially when one begins to have to consider permutations with different types of unpairables. For example, we may have to consider permutations with one number group of the type described above, where the number group has two pairs of elements which only pair with one another, but also another number group, which has an unpairable.

It may yet be that this method will prevail, and find the P -recursivity of $H_n(4)$. However, we begin to suspect, due to the number of different possible block types, or equivalently, the different possible types of rows and columns in a magic square of line sum 4, that the number of magic squares of line sum 4 is not P -recursive. This would be unfortunate, considering that it is almost impossible to show that a given sequence is not P -recursive.

Another possible direction of continued research on magic squares would be to find the degree of the recursions. We know, from Chapter 2, that the degree of the recursion of $H_n(2)$ is two. One may now ask the same question about $H_n(3)$. In this dissertation, we have shown that $H_n(3)$ is P -recursive. We have been unable to find the degree of said recurrence.

Chapter 3 describes an entirely new approach to magic cubes. As we have seen, not much is known about these objects, and as such, this is an excellent area for continued development and research. From here, we hope to find more conditions under which a magic cube can be completed. Perhaps completion questions will lead to an upper bound on the number of magic cubes of a given line sum, as it once did for Latin Squares of given size. For this connection between completion of a Latin Square and their asymptotic number, see [7].

Finally, for one other direction of possible research: Is there a way to define a "Double Magic Cube" which would aid us in the enumeration of magic cubes of line sum 2. Perhaps this would give a method of expressing the number of such magic cubes in terms of the number of Latin Squares. While the number

of Latin Squares is not known, this would at least lead to an answer on the asymptotic behavior of magic cubes. This idea seems rather difficult, however. The complication enters the argument with irreducible magic cubes. As these do not reduce to sums of magic cubes of line sum 1, the argument for a relation with any defined "Double Magic Cube" will most likely require special consideration for the irreducible magic cubes. As of the time of this writing, we have not been able to overcome this difficulty.

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BIOGRAPHICAL SKETCH

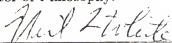
I was born in Panama, while my father was stationed at the Canal. My early years were spent in many different locales, and I have lived all over the eastern seaboard of the United States. I had an early interest in mathematics, and have received bachelor's and master's degrees from the University of Florida.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.



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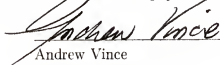
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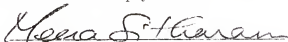
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May 2004

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ON INTEGER SOLUTIONS TO SYSTEMS OF LINEAR EQUATIONS

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Degree: Doctor of Philosophy

Graduation Date: May 2004

The subject of combinatorics is an extensive one. One of the questions we ask is, for a given object, "how many are there?" In this work, we answer this question for what we call magic squares. Using the method contained herein, it is possible to express these objects in a new way, which leads to easier enumeration. By offering a different outlook on magic squares, we hope to lead the mathematical community closer to a final answer on this question.